Section A: Pure Mathematics

1 Use the binomial expansion to obtain a polynomial of degree 2 which is a good approximation to $\surd(1-x)$ when x is small.

(i) By taking $x = 1/100$, show that $\surd(11) \approx 79599/24000$, and estimate, correct to 1 significant figure, the error in this approximation. (You may assume that the error is given approximately by the first neglected term in the binomial expansion.)

(ii) Find a rational number which approximates $\surd(1111)$ with an error of about 2×10^{-12}.

2 Sketch the graph of the function $[x/N]$, for $0 < x < 2N$, where the notation $[y]$ means the integer part of y. (Thus $[2.9] = 2$, $[4] = 4$.)

(i) Prove that

$$\sum_{k=1}^{2N}(-1)^{[k/N]}k = 2N - N^2.$$

(ii) Let

$$S_N = \sum_{k=1}^{2N}(-1)^{[k/N]}2^{-k}.$$

Find S_N in terms of N and determine the limit of S_N as $N \to \infty$.

3 The cuboid $ABCDEFGH$ is such AE, BF, CG, DH are perpendicular to the opposite faces $ABCD$ and $EFGH$, and $AB = 2, BC = 1, AE = \lambda$. Show that if α is the acute angle between the diagonals AG and BH then

$$\cos\alpha = \left|\frac{3-\lambda^2}{5+\lambda^2}\right|$$

Let R be the ratio of the volume of the cuboid to its surface area. Show that $R < 1/3$ for all possible values of λ.

Prove that, if $R \geqslant 1/4$, then $\alpha \leqslant \arccos(1/9)$.

4 Let

$$f(x) = P\sin x + Q\sin 2x + R\sin 3x\ .$$

Show that if $Q^2 < 4R(P-R)$, then the only values of x for which $f(x) = 0$ are given by $x = m\pi$, where m is an integer.
[You may assume that $\sin 3x = \sin x(4\cos^2 x - 1)$.]

Now let

$$g(x) = \sin 2nx + \sin 4nx - \sin 6nx,$$

where n is a positive integer and $0 < x < \pi/2$. Find an expression for the largest root of the equation $g(x) = 0$, distinguishing between the cases where n is even and n is odd.

5 The curve C_1 passes through the origin in the x–y plane and its gradient is given by

$$\frac{dy}{dx} = x(1-x^2)e^{-x^2}.$$

Show that C_1 has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}e^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of C_1.

The curve C_2 passes through the origin and its gradient is given by

$$\frac{dy}{dx} = x(1-x^2)e^{-x^3}.$$

Show that C_2 has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}e^{-1}$. (You need not find k.)

6 Show that

$$\int_0^1 \frac{x^4}{1+x^2}\,dx = \frac{\pi}{4} - \frac{2}{3}\ .$$

Determine the values of

(i) $\displaystyle\int_0^1 x^3\ \tan^{-1}\left(\frac{1-x}{1+x}\right)\ dx\ ,$

(ii) $\displaystyle\int_0^1 \frac{(1-y)^3}{(1+y)^5}\tan^{-1}y\ dy\ .$

7 In an Argand diagram, O is the origin and P is the point $2+0i$. The points Q, R and S are such that the lengths OP, PQ, QR and RS are all equal, and the angles OPQ, PQR and QRS are all equal to $5\pi/6$, so that the points O, P, Q, R and S are five vertices of a regular 12-sided polygon lying in the upper half of the Argand diagram. Show that Q is the point $2+\sqrt{3}+i$ and find S.

The point C is the centre of the circle that passes through the points O, P and Q. Show that, if the polygon is rotated anticlockwise about O until C first lies on the real axis, the new position of S is

$$-\frac{1}{2}(3\sqrt{2}+\sqrt{6})(\sqrt{3}-i)\ .$$

8 The function f satisfies $\mathrm{f}(x+1)=\mathrm{f}(x)$ and $\mathrm{f}(x)>0$ for all x.

(i) Give an example of such a function.

(ii) The function F satisfies

$$\frac{\mathrm{dF}}{\mathrm{d}x}=\mathrm{f}(x)$$

and $\mathrm{F}(0)=0$. Show that $\mathrm{F}(n)=n\mathrm{F}(1)$, for any positive integer n.

(iii) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{f}(x)y=0$$

that satisfies $y=1$ when $x=0$. Show that $y(n)\to 0$ as $n\to\infty$, where $n=1,\,2,\,3,\,\ldots$

Section B: Mechanics

9 A particle of unit mass is projected vertically upwards with speed u. At height x, while the particle is moving upwards, it is found to experience a total force F, due to gravity and air resistance, given by $F = \alpha\,\mathrm{e}^{-\beta x}$, where α and β are positive constants. Calculate the energy expended in reaching this height. Show that

$$F = \tfrac{1}{2}\beta v^2 + \alpha - \tfrac{1}{2}\beta u^2 \ ,$$

where v is the speed of the particle, and explain why $\alpha = \frac{1}{2}\beta u^2 + g$, where g is the acceleration due to gravity.

Determine an expression, in terms of y, g and β, for the air resistance experienced by the particle on its downward journey when it is at a distance y below its highest point.

10 Two particles A and B of masses m and km, respectively, are at rest on a smooth horizontal surface. The direction of the line passing through A and B is perpendicular to a vertical wall which is on the other side of B from A. The particle A is now set in motion towards B with speed u. The coefficient of restitution between A and B is e_1 and between B and the wall is e_2. Show that there will be a second collision between A and B provided

$$k < \frac{1 + e_2(1 + e_1)}{e_1} \ .$$

Show that, if $e_1 = \frac{1}{3}$, $e_2 = \frac{1}{2}$ and $k < 5$, then the kinetic energy of A and B immediately after B rebounds from the wall is greater than $mu^2/27$.

11 A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity. Show that P will hit the ground at a distance R from A given by

$$\frac{gR}{u} = v + v_e + \sqrt{(v_e^2 + v^2)}.$$

It is required that the range R should be greater than a certain distance D (where $D > 2uv/g$). Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[*The effect of air resistance is to be neglected.*]

Section C: Probability and Statistics

12 The national lottery of Ruritania is based on the positive integers from 1 to N, where N is very large and fixed. Tickets cost £1 each. For each ticket purchased, the punter (i.e. the purchaser) chooses a number from 1 to N. The winning number is chosen at random, and the jackpot is shared equally amongst those punters who chose the winning number.

A syndicate decides to buy N tickets, choosing every number once to be sure of winning a share of the jackpot. The total number of tickets purchased in this draw is $3.8N$ and the jackpot is £W. Assuming that the non-syndicate punters choose their numbers independently and at random, find the most probable number of winning tickets and show that the expected net loss of the syndicate is approximately

$$N \;-\; \frac{5(1-e^{-2.8})}{14}\,W\;.$$

13 The life times of a large batch of electric light bulbs are independently and identically distributed. The probability that the life time, T hours, of a given light bulb is greater than t hours is given by

$$\mathrm{P}(T>t) \;=\; \frac{1}{(1+kt)^{\alpha}}\;,$$

where α and k are constants, and $\alpha > 1$. Find the median M and the mean m of T in terms of α and k.

Nine randomly selected bulbs are switched on simultaneously and are left until all have failed. The fifth failure occurs at 1000 hours and the mean life time of all the bulbs is found to be 2400 hours. Show that $\alpha \approx 2$ and find the approximate value of k. Hence estimate the probability that, if a randomly selected bulb is found to last M hours, it will last a further $m - M$ hours.

14 Two coins A and B are tossed together. A has probability p of showing a head, and B has probability $2p$, independent of A, of showing a head, where $0 < p < \frac{1}{2}$. The random variable X takes the value 1 if A shows a head and it takes the value 0 if A shows a tail. The random variable Y takes the value 1 if B shows a head and it takes the value 0 if B shows a tail. The random variable T is defined by

$$T = \lambda X + \tfrac{1}{2}(1 - \lambda)Y.$$

Show that $\mathrm{E}(T) = p$ and find an expression for $\mathrm{Var}\,(T)$ in terms of p and λ. Show that as λ varies, the minimum of $\mathrm{Var}\,(T)$ occurs when

$$\lambda = \frac{1 - 2p}{3 - 4p}\,.$$

The two coins are tossed n times, where $n > 30$, and $\overline{T}$ is the mean value of T. Let b be a fixed positive number. Show that the maximum value of $\mathrm{P}(|\overline{T} - p| < b)$ as λ varies is approximately $2\Phi(b/s) - 1$, where Φ is the cumulative distribution function of a standard normal variate and

$$s^2 = \frac{p(1-p)(1-2p)}{(3-4p)n}\,.$$